1. Use structural induction to prove that a graph with maximum degree d can be properly vertex-colored using d+1 colors

Base case: a single vertex that has degree = 0 needs 1 color to be properly vertex-colored.

Inductive Hypothesis: Any graph with at least two vertices and a maximum degree d can be properly vertex-colored using at most d+1 colors.

Let v be a vertex that has a maximum degree in the graph G that’s properly vertex-colored. If we were to remove v from the graph then the rest of the graph could be colored with d colors by the inductive hypothesis. Now if we were to add a vertex in its place, it would have to be a different color than any of the other vertices it would be adjacent to. So, we would need at least d+1 colors to properly vertex color the entire graph since we just added a vertex with a new color.

2. Prove that for any graph G, the product of its chromatic number and its independence number is at least the number of vertices in G.

Base case: A graph with one vertex has an independent value of one and a chromatic number of one, so the sum would be two. There is only one vertex in the graph so the number of vertices is less than the above sum.

Inductive Hypothesis: For any graph with at least two vertices will have the sum of the chromatic number and the independent value are at least the number of vertices in the graph.

Let’s assume we have a graph with a chromatic number of n and an independence number of k, with a total number of vertices v. Then, by the inductive hypothesis n + k >= v. We have two cases in which adding a vertex would affect the graph, either adding an isolated vertex or adding a vertex with an at least one edge to the connected vertices. If we add an isolated vertex, then we increase the independence number by one. Thus we would have n + (k + 1) > = v + 1 and then by arithmetic we would have n + k > = v.

In the other case, we add a vertex that has at least one edge to another vertex. This would lead to either increasing the independence number by the amount v minus the number of edges, or increasing the chromatic number if the degree is the highest in the graph (Using the proof in problem one). This would lead to an overall increase in either n or k by some constant L where L is a positive integer greater than 1. Then we would have n + k + L >= v + 1. We can assume that L is the minimum number of 1, so that we have n + k + 1 >= v +1 which is the same as above.